Lecture 2

Loose ends about Grassmannians

(1) $W \subset V$ $\quad \dim W = d \quad \dim V = n$

Then $i_W : W \hookrightarrow V$ induces $(i_W)_* : \Lambda^d W \to \Lambda^d V$.

And $\Lambda^d W \cong k$.

So Plücker embedding is just $\quad \begin{aligned} Gr(d,V) &\to \Lambda^d V \\ W &\longmapsto \text{image } (i_W)_* \end{aligned}$

(Nice that the inverse, where defined, is the kernel of a linear map naturally assoc to any line in $\Lambda^d V$.)

(2) Tangent space.

Given $W \in Gr(d,V)$, we have a chart identifying a nbd of W with $\text{Hom}(W,Z)$ for any complement Z of W.

So $T_W Gr(d,V) \cong \text{Hom}(W,Z)$ non-canonically.

But $T_W Gr(d,V) \cong \text{Hom}(W, V/W)$ <u>canonically</u>.

E.g. if Z, Z' are two complements of V, then

$$\begin{tikzcd}
\text{Hom}(W,Z') \arrow[dr, "{\substack{Z' \to V/W \\ \text{ind by } Z' \hookrightarrow V}}"'] & T_W Gr(d,V) \arrow[l, "\sim \text{chart}"'] \arrow[r, "\sim \text{chart}"] \arrow[d, "*"] & \text{Hom}(W,Z) \arrow[dl, "{\substack{Z \to V/W \\ \text{ind by } Z \hookrightarrow V}}"] \\
& \text{Hom}(W, V/W) &
\end{tikzcd}$$

they both "agree" with the canonical one using the isomorphisms $Z, Z' \to V/W$ induced by the inclusions $Z, Z' \hookrightarrow V$.

How to define $*$? Let's define its inverse.

$$\varphi \in \text{Hom}(W, V/W) \longmapsto \gamma_\varphi'(0) \qquad \gamma_\varphi : (-\varepsilon, \varepsilon) \to Gr(d,V)$$

Defined by $\gamma_\psi(t) = \text{image}(i_W + t\tilde{\psi})$

$i_W : W \hookrightarrow V$ and $\tilde{\psi} : W \to V$ is any lift of ψ. (Check lift doesn't matter!)

In particular, $T_\ell \mathbb{P}(V) \cong \text{Hom}(\ell, V/\ell)$ (perhaps more familiar?)

(3) Notation $Gr(d,n)$ or $Gr_k(d,n)$ means $Gr(d,k^n)$.

e.g. $Gr_{\mathbb{R}}(2,4)$ $Gr_{\mathbb{C}}(2,4)$

Flag varieties / Flag manifolds

A (partial) flag in V is a tuple of subspaces $(F_0, \dots, F_m)$ where $0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_m = V$. Can omit F_0, F_m, or both (unambiguously!).

The dimension of a flag is $(\dim F_1, \dots, \dim F_{m-1})$. An increasing sequence of positive integers less than n.

This dimension seems natural but turns out not to be the most useful quantity in many cases. Instead we often need the equivalent data:

The type of a flag is $(d_1, \dots, d_m)$ where $d_i = \dim(F_i/F_{i-1})$.

So $\sum_i d_i = n$ and $d_i > 0$. Any such tuple is possible.

Let $\text{Flag}(V, d_1, \dots, d_m)$ denote the set of all flags of type $(d_1, \dots, d_m)$.

(Shorthand: $\text{Flag}(d_1, \dots, d_m) = \text{Flag}(k^{\sum d_i}, d_1, \dots, d_m)$)

E.g. $\mathrm{Flag}(V, 1, n-1) = \mathbb{P}(V)$

$\mathrm{Flag}(V, n-1, 1) = \mathbb{P}^*(V)$

$\mathrm{Flag}(V, d, n-d) = \mathrm{Gr}(d, V)$

$\mathrm{Flag}(V, 1, \dots, 1)$ is the set of full flags ($\dim F_i = i$, $m = n$) and is abbreviated $\mathrm{Flag}(V)$.

Theorem. $\mathrm{Flag}(V, d_1, \dots, d_m)$ is a compact smooth manifold of dimension $\beta \sum_{1 \le i \le j \le m} d_i d_j$. It is also a smooth projective k-alg variety.

There are obvious maps
$$\mathrm{Flag}(V, d_1, \dots, d_m) \to \mathrm{Gr}(d_1 + \dots + d_i, V)$$
$$F \longmapsto F_i$$
and taking the product of them we do get an embedding
$$\mathrm{Flag}(\cdots) \longrightarrow \prod_i \mathrm{Gr}(\cdots, V) \longrightarrow \prod_i \mathbb{P}(\Lambda^{\cdots} V)$$
and one approach is to study this map & its image.

Instead, we'll look at $\mathrm{Flag}(V, \underline{d})$ as a homogeneous space.

Obs. $\mathrm{Aut}(V)$ acts transitively on $\mathrm{Flag}(V, \underline{d})$ by $A \in \mathrm{Aut}(V)$
$$(F_1, \dots, F_m) \xmapsto[A]{} (A(F_1), \dots, A(F_m)).$$

Pf. Let's do the case of full flags.

$(F_1, \dots, F_n) \rightsquigarrow$ there is a basis $(f_1, \dots, f_n)$ s.t. $F_i = \mathrm{span}(f_1, \dots, f_i)$

$(G_1, \dots, G_n) \rightsquigarrow \cdots (g_1, \dots, g_n)$

Any two bases are related by an automorphism, $A(f_i) = g_i$

Then $A \cdot F = G$. $\square$ Easy exercise: generalize to partial flags.

Terminology: A basis $(f_1, \dots, f_n)$ adapted to F as above will be called a <u>flag basis</u> for F.

Now, let's recall a couple of generalities about **G-spaces**. G an abstract group. (i.e. spaces w/ an action of G).

If $G \curvearrowright X$ transitively, $x_0 \in X$, and $H = \mathrm{Stab}(x_0) = \{g \mid g \cdot x_0 = x_0\}$

then the map
$$\begin{aligned} G/H &\longrightarrow X \\ gH &\longmapsto g \cdot x_0 \end{aligned}$$
is a well-defined G-equivariant bijection, where the action of G on G/H is
$$a \cdot (gH) = agH.$$
(This is a case of the orbit-stabilizer theorem.)

So take $F_0 \in \mathrm{Flag}(V, \underline{d})$ as a base point. We get
$$\mathrm{Flag}(V, \underline{d}) \underset{\text{bij}}{\cong} \mathrm{Aut}(V)/P$$
an $\mathrm{Aut}(V)$-equivariant bijection.

if $g_n \to g$ and $g_n \cdot F_i = F_i$ then $g \cdot F_i = F_i$

Now $\mathrm{Aut}(V) \cong GL_n(k)$ is a Lie group and P is a $\boxed{\text{closed subgroup}}$, hence P is an embedded Lie subgrp.

<u>Theorem</u> (Quotient mfld theorem, transitive case) Let G be a Lie group and H an embedded Lie subgroup. Then G/H has a unique smooth structure of dimension $\dim G - \dim H$ s.t. the natural map
$$G \xrightarrow{\pi} G/H$$
is a submersion. In particular, $\ker d\pi \cong T_e H = \mathfrak{h}$ and $T_{eH} G/H \cong \underbrace{\mathfrak{g}/\mathfrak{h}}_{\text{VECTOR SPACE ONLY!}}$ where $\mathfrak{g} = T_e G = $ Lie alg of G.

Question: Should I say anything about this theorem?
It is [Lee 21.17] but the real action is [Lee 21.10]

Cor: If $Z \subset \mathfrak{g}$ is a subspace complementary to $\mathfrak{h}$, then $d\pi|_Z : Z \to T_{eH} G/H$ is an isomorphism.

So we obtain a smooth structure on $\mathrm{Flag}(V, \underline{d})$ of dimension $\dim(\mathrm{Aut}(V)) - \dim(P)$.

What is P? Let's start with full flags

Let $f_1 \dots f_n$ be a flag basis of $F_0 \in \mathrm{Flag}(V)$.

By taking matrices relative to $f_1 .. f_n$ we get

$$\mathrm{Aut}(V) \xrightarrow{\sim} GL_n(k). \qquad \text{(Lie group iso.)}$$

So it suffices to describe the corresp. subgroup of $GL_n(k)$.

Lem. $g \in \mathrm{Aut}(V)$ preserves F_0 iff its matrix relative to $(f_1 \dots f_n)$ has zeros below the diagonal.

i.e. $A = [g]_{f_1 \dots f_n} = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}$

Pf. Exercise. The zeros in column j mean $(F_0)_j = \mathrm{span}(f_1 .. f_j)$ is invariant. ◻

So in this case $P \cong$ invertible matrices w/ zero below diag

$T_e P \cong \mathfrak{p}$ = matrices w/ zero below diag

$$\dim P = n + (n-1) + \cdots + 1 = \frac{n(n+1)}{2}.$$

$$\dim \mathrm{Flag}(V) = \dim \mathrm{Aut}(V) - \dim P = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

$= \dim Z$ where $Z = \begin{pmatrix} 0 & \cdots & 0 \\ * & \ddots & \vdots \\ * & \cdots * & 0 \end{pmatrix} \subset \mathfrak{gl}_n k$